

Phase transition via entanglement entropy in AdS_3/CFT_2 superconductors

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The purpose of this report is to provide a framework for defining phase transition processes in two dimensional holographic superconductors, and to illustrate how they are useful to be described by holographic entanglement entropy. We study holographic entanglement entropy in a two dimensional fully back-reacted model for holographic superconductors. We prove that phase transition could be observe using a discontinuity in the first order of entropy.

Keywords: AdS/CFT Correspondence; Holography and condensed matter physics (AdS/CMT); Entanglement entropy.

1. Introduction

Entanglement lies at the heart of many aspects of quantum information theory and it is therefore desirable to understand its structure as well as possible. One attempt to improve our understanding of entanglement is the study of our ability to perform information theoretic tasks locally based on Anti-de Sitter space/Conformal Field Theory (AdS/CFT) conjecture¹, such as the strongly coupling, like type II superconductors²⁻³. holographic superconductors, Quark-Gluon plasma, and superconductor/superfluid in condensed matter physics, particularly using qualitative approaches⁴⁻⁶. In^{7,8} it was shown that entanglement entropy of any entangled CFT system could be calculated using the area of the minimal surfaces in bulk. This idea called holographic entanglement entropy (HEE) and finds a lot of applications⁹⁻²⁴, as a geometric approach. In order to address this issues, we consider two possible portions \hat{A} (set A), $B = \hat{A}'$ (the complementary set) of a single quantum system. We make a complete Hilbert space $\mathcal{H}_{\hat{A}} \times \mathcal{H}_{\hat{A}'}$.

Let us to consider the Von-Neumann entropy $S_X = -Tr_X(\rho \log \rho)$, here Tr is the quantum trace of quantum operator ρ over quantum basis X . If we compute $S_{\tilde{A}}$ and $S_{\tilde{A}'}$, we obtain $S_{\tilde{A}} = S_{\tilde{A}'}$. A physical meaning is that Von-Neumann entropies are now more likely to identify it with the surface boundary area $\partial \tilde{A}$ ²⁵. Following to the proposal^{7,8}, every quantum field theory in (d) dimensional flat boundary has a gravitational dual embedded in AdS_{d+1} bulk. The holographic algorithm tells us that the HEE of a region of space \tilde{A} and its complement from the AdS_{d+1} geometry of bulk:

$$S_{\tilde{A}} \equiv S_{HEE} = \frac{Area(\gamma_{\tilde{A}})}{4G_{d+1}}, \quad (1)$$

We should firstly find the minimal $(d-1)$ D mini-super surface $\gamma_{\tilde{A}}$. It had been assumed to extend $\gamma_{\tilde{A}}|_{AdS_{d+1}}$ to bulk, but with criteria to keep surfaces with same boundary $\partial \gamma_{\tilde{A}}$ and $\partial \tilde{A}$. Computation of the HEE have been initiated²⁶⁻²⁸.

Another surprising development is that HEE can be used to describe phase transitions in CFT systems. Here we discuss this issue of observing phase transition in a two dimensional holographic superconductor, and show that in this approach the first order phase transition can be detected.

2. Model for 2D HSC

A toy model for two dimensional holographic superconductors (2DHSC) using the AdS_3/CFT_2 proposed by the following action²⁹⁻³⁴:

$$S = \int d^3x \sqrt{-g} \left[\frac{1}{2\kappa^2} (R + \frac{2}{L^2}) - \frac{1}{4} F^{ab} F_{ab} - |\nabla \phi - iA\phi|^2 - m^2 |\phi|^2 \right]. \quad (2)$$

Here, $\kappa^2 = 8\pi G_3$ defines the three dimensional gravitational constant, the Newton constant G_3 , L is the AdS radius, $m^2 = m_\phi^2 \in (-1, \infty)$ mass of scalar field, and $g = \det(g_{\mu\nu})$.

We adopt the following set of planar coordinates for asymptotically AdS_3 spacetime:

$$ds^2 = -f(r)e^{-\beta(r)} dt^2 + \frac{d\gamma^2}{f(r)} + \frac{r^2}{L^2} dx^2. \quad (3)$$

To have thermal behavior we choose the temperature for CFT_2 from our AdS_3 black hole following Bekenstein-Hawking formula:

$$T = \frac{f'(r_+)e^{-\beta(r_+)/2}}{4\pi}. \quad (4)$$

here r_+ is the horizon the the blackhole , the largest root of the Eq. $f(r_+) = 0$. To preserve staticity, we select the that the Abelian gauge field A_μ and scalar field ϕ are static (time independent) and spherically symmetric :

$$A_t = A(r)dt, \quad \phi \equiv \phi(r). \quad (5)$$

It is useful to write the set of the equations of motion (EOM)s in terms of the new radial coordinate $z = \frac{r_+}{r}$,. Using this coordinate the black hole horizon shifted to the $z = 1$ and AdS boundary locates at $z = 0$. We need to study solutions of the following set of non linear ordinary differential Eqs. near $z = 0$:

$$\phi'' + \frac{\phi'}{z} \left[1 + \frac{zf'}{f} - \frac{z\beta'}{2} \right] + \frac{r_+^2 \phi}{z^4} \left[\frac{A^2 e^\beta}{f^2} - \frac{m^2}{f} \right] = 0, \quad (6)$$

$$A'' + \frac{A'}{z} \left[1 - \frac{z\beta'}{2} \right] - \frac{2r_+^2 A \phi^2}{z^4 f} = 0, \quad (7)$$

$$\beta' - \frac{4\kappa^2 r_+^2}{z^3} \left[\frac{A^2 \phi^2 e^\beta}{f^2} - \frac{z^4 \phi'^2}{r_+^2} \right] = 0, \quad (8)$$

$$f' - \frac{2r_+^2}{L^2 z^3} - \kappa^2 z e^\beta A'^2 - \frac{2\kappa^2 m^2 r_+^2 \phi^2}{z^3} - \frac{2\kappa^2 r_+^2}{z^3} \left[\frac{A^2 \phi^2 e^\beta}{f} + \frac{f \phi'^2 z^4}{r_+^2} \right] = 0, \quad (9)$$

The numerical and analytical solutions of the above Eqs. with $\phi \neq 0, T < T_c$ were presented in Ref.^{29,30}.

3. Calculation of holographic entanglement entropy

We need to specify the bulk system to find minimal area surfaces. The appropriate systems parametrization id to represent bulk sector in one degree of freedom $\tilde{A} := \{t = t_0, -\theta_0 \leq \theta \leq \theta_0, r = r(\theta)\}$. Using metric given in Eq. (3), the total length (angle) θ_0 and HEE are defined by the following integrals:

$$\theta_0 = \int_0^{\theta_0} d\theta = \int_0^{\theta_0} \frac{C dr}{r \sqrt{f(r)(r^2 - C^2)}} \quad (10)$$

$$S_{HEE} \equiv \frac{1}{2G_3} \int_0^{\theta_0} \frac{r dr}{\sqrt{f(r)(r^2 - C^2)}} \quad (11)$$

We need to evaluate the sensitivity of (10,11) in the bulk of acute regimes of temperature. We apply the domain wall approximation analysis³⁵ which is based on the idea is proposed to investigate some aspects of the HEE

along renormalization group (RG) trajectories. The AdS_3 is relating to RG flows in $(1+1)D$.^a The geometry (metric) which we will use is called here domain wall geometry, in which we suppose that two parts of the AdS are connected by a wall with two AdS radius $L_{IR} > L_{UV}$. Furthermore we suppose that the length scale of AdS L , is defined as the following:

$$L = \begin{cases} L_{UV} , & r > r_{DW} \\ L_{IR} , & r < r_{DW} \end{cases} . \quad (12)$$

We have a sharp phase transition between two patches of the AdS space time, so we locate the DW at the radius $r = r_{DW}$. In this approach we define r_* as a possible “turning” point of the minimal surface $\gamma_{\bar{A}}$. It is defined by $r'(\theta)|_{r=r_*} = 0$. We will suppose that $r_* < r_{DW}$. We replaced the integrating out to $r = +\infty$ by integrating out to large positive radius r_{UV} . Indeed, we assume that r_{UV} stands out for UV cutoff²⁸. We can rewrite HEE as we like:

$$S_{HEE} = \frac{1}{2G_3} [S_{IR} + S_{UV}] , \quad (13)$$

$$S_{IR} = \int_{r_*}^{r_{DW}} \frac{r dr}{\sqrt{f_{IR}(r^2 - L_{IR}^2)}} , \quad (14)$$

$$S_{UV} = \int_{r_{DW}}^{r_{UV}} \frac{r dr}{\sqrt{f_{UV}(r^2 - L_{UV}^2)}} . \quad (15)$$

In both cases IR,UV, the geometry of AdS has imposed tight constraints on the metric. Furthermore the angle is defined by a two partion integral, θ_{IR}, θ_{UV} :

$$\theta_{IR} = i \log \left[\frac{r_*}{r_{DW}} \frac{\sqrt{L_{IR}^2 - r_{DW}^2} - L_{IR}}{\sqrt{L_{IR}^2 - r_*^2} - L_{IR}} \right] , \quad (16)$$

$$\theta_{UV} = \frac{\sqrt{r_{UV}^2 - L^2}}{r_{UV}} - \frac{\sqrt{r_{DW}^2 - L^2}}{r_{DW}} . \quad (17)$$

The entanglement entropy can be computed as the following form:

$$S_{IR} = \sqrt{r_{DW}^2 - L_{IR}^2} - \sqrt{r_*^2 - L_{IR}^2} \quad (18)$$

$$S_{UV} = \frac{iL^2}{L_{UV}} \log \left[\frac{r_{DW}}{r_{UV}} \frac{\sqrt{L_{UV}^2 - r_{UV}^2} - L_{UV}}{\sqrt{L_{UV}^2 - r_{DW}^2} - L_{UV}} \right] \quad (19)$$

^aThe RG flow is defined as the $N = 1$ SUSY deformation of $N = 4$ SUSY-YM theory.

4. HEE close to the $T \lesssim T_c$

The normal phase of the system can even be achieved when we set the scalar field $\phi = 0$. In this case the EE between \tilde{A} and its complement is given by:

$$s_{\tilde{A}} = 4G_3 S_{HEE} = 2r_*^{-1} \int_{r_{UV}}^{r_*} \frac{r dr}{\sqrt{f(r)(r^2 - r_*^2)}} \quad (20)$$

We rewrite $s_{\tilde{A}}$ in terms of the coordinate z :

$$s_{\tilde{A}} = 2r_+ r_* \int_{z_{UV}}^{z_*} \frac{dz}{z^3 \sqrt{f(z)} \sqrt{z^{-2} - z_*^{-2}}}. \quad (21)$$

Near the critical point $T \lesssim T_c$ we can approximate the integral as the following:

$$s_{\tilde{A}} = 2r_{+c} r_* \int_{z_{UV}}^{z_*} \frac{dz}{z^3 \sqrt{f_0} \sqrt{z^{-2} - z_*^{-2}}}. \quad (22)$$

and for the angle,

$$\begin{aligned} \frac{\theta_0}{2} &= r_* \int_{r_{UV}}^{r_*} \frac{dr}{r \sqrt{f(r)(r^2 - r_*^2)}} \\ &= \frac{r_*}{r_+} \int_{z_{UV}}^{z_*} \frac{dz}{z \sqrt{f(z)(z^{-2} - z_*^{-2})}}. \end{aligned} \quad (23)$$

At criticality when $T \lesssim T_c$ we obtain:

$$\frac{\theta_0}{2} = \frac{r_*}{r_{+c}} \int_{z_{UV}}^{z_*} \frac{dz}{z \sqrt{f_0(z^{-2} - z_*^{-2})}}. \quad (24)$$

where

$$T_c = \frac{1}{4\pi r_{+c}} \left(2r_{+c}^2 L^{-2} - \kappa^2 \mu_c^2 \right) \quad (25)$$

We evaluate the HEE and angle(length) numerically with $T_c = 0.01$. Numeric analysis showed³⁸:

- When we increase the temperature and dual chemical potential, HEE is also increasing smoothly. System becomes a normal conductor at high temperatures and tends to a superconductor phase when temperature decreases.

- If we consider HEE in constant temperature (isothermal curves), we showed that there exists at least one lower temperature regime $T < T_c$ is almost compulsory for superconductivity. Furthermore, we conclude that there is no "confinement/deconfinement" phase transition point in the CFT_2 .
- If we fixed the relative chemical potential $\frac{\mu}{\mu_c}$, we observe that by increasing temperature we have much more amount of HEE. The slope of the HEE $\frac{dS}{dT}$ decreases as the relative chemical potential $\frac{\mu}{\mu_c} \neq 1$ decreases. We interpret this phenomena as emergent of more degrees of freedom (dof) at low temperature regimes.
- Both of the HEE and the characteristic length of the entangled system are always increasing with respect to the temperature T , and never decreasing for fixed T_c .
- HEE is linear function of length. One reason is that in small values of belt angle (small sizes) the system emerges new extra dof. Another reason could be understood via first law of entanglement entropy^{39, 40}.
- When we add the scalar field to the system, $\phi(z) \neq 0$, if we adjust data as $\epsilon_0^2 = 0.05, \kappa\mu_c = 0.005$, we found the critical temperature $T_c = 0.2$. It was demonstrated that the system softly has transition from normal phase $T > T_c$ to the superconductor phase $T \lesssim T_c$ for $T \approx 0.0179, 0.0173, 0.0165, 0.0152, 0.0132$.
- We demonstrated that the slope of the HEE with respect to the belt angle $\frac{ds'}{d\theta'}$ remains constant. It means that there is no critical belt length θ_c in our CFT_2 system.

5. First order phase transition at critical point

A numerical study of HEE and length shows that these quantities were smooth, and that their behaviors went most smoothly when the phase transition held the same mechanism as the usual. When the temperature of the system T rattends to its critical value T_c , we detect a discontinuity in the $\frac{dS}{dT}$ when the phase is changed, and a first order phase transition may be introduced into the system. By computing the difference between $\frac{dS}{dT}$ for $T > T_c$ and $T < T_c$ at $T = T_c$ (scaled the entropy in log scale) we observe that this phase transition is of the first order. Indeed, at the critical point $\lim_{T \rightarrow T_c} \frac{dS}{dT} = \infty$ and we observe the first order phase transitions from the behavior of the entanglement entropy HEE at the critical point $T = T_c$. These types of first order phase transitions have been observed recently in

literature⁴⁰. We conclude that the HEE is indeed a good probe to phase transition in lower dimensional holographic superconductors. Furthermore, it implies that the HEE can indicate not only the occurrence of the phase transition, but also we can learn about the order of the phase transition from it.

6. Summary

Prior work has documented the effectiveness of holographic entanglement entropy in describing phase transition of quantum systems. Liu et al.²⁹ for example, reports that holographic superconductors in two dimensional systems moving towards the phase transition in lower dimensional quantum field theory systems improved across several studies. However, these studies have either been purely bulk studies or have not focused on thermodynamic whose entanglement entropy was boundary-related. In this study we showed that entanglement entropy predicts a first order phase transition in two dimensional holographic superconductors.

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